Control of hyperchaos

Shouliang Bu, Shaoqing Wang, and Hengqiang Ye
Laboratory of Atomic Imaging of Solids, Institute of Metal Research, Chinese Academy of Sciences, Shenyang 110016, China
(Received 29 May 2001; published 22 October 2001)

A general method for controlling chaotic systems with one or more positive Lyapunov exponents is investigated analytically and numerically. The method retains the formal features of the adaptive adjustment mechanism and can be equally applied to various types of the unstable fixed points. It is shown that the method proposed here neither asks for any prior analytical knowledge of the system, nor any internal or external controlling parameters in advance.

DOI: 10.1103/PhysRevE.64.056212 PACS number(s): 05.45.Gg, 05.45.Jn, 05.45.Pq

It is well known that there is an infinite number of unstable periodic orbits (UPOs) embedded in the chaotic attractors [1–3]. How to stabilize UPOs, i.e., control chaos, is now an active topic in nonlinear sciences and has received much attention in the last decade [4–13]. Since the original work of Ott, Grebogi and York (OGY) [4], many different controlling methods have been proposed and are being pursued [5–15]. However, most of these methods are only appropriate for low dimensional chaotic systems (here, we refer to the system with one positive Lyapunov exponent). In this situation, there is just one unstable direction along the chaotic orbit. The problem of how to stabilize high dimensional chaotic systems, the systems with multiple positive Lyapunov exponents, i.e., the so-called hyperchaos, is considered to be a tough one.

Recently, Yang, Liu, and Mao use a new remarkable method (we refer to the YLM method hereafter) to control chaotic systems [14]. This method doesnot require preexistence of a stable manifold and can be applied to control hyperchaotic systems efficiently. Similar to the OGY method and its variants, the YLM method is also based on a parameter perturbations mechanism and so requires one to find at least one adjustable controlling parameter of the system in advance. For the YLM method, by adjusting the corresponding control parameter, one of the unstable directions becomes stable so as to stabilize the unstable orbit. Unfortunately, in many real systems such as biological, chemical, and social economical systems, one often cannot find such a parameter at all. Our approach to overcome this shortcoming is the adaptive adjustment mechanism (AAM). The AAM takes advantage of the variable feedback control and doesnot require existence of adjustable controlling parameters. However, the AAM can only be applied definitely to control some special types of fixed points. Even if one utilizes the so-called nonuniform AAM [15], there also exist some situations to which the AAM or nonuniform AAM cannot be applied. According to Ref. [15], the fixed points are classified into four types. If an unstable fixed point is either a type-I fixed point ($a_j<1$ for all $j=1,2,...,n$) or a type-II fixed point ($a_j>1$ for all $j=1,2,...,n$), it can always be stabilized by the AAM or nonuniform AAM. Here, a_j ($j=1,2,...,n$) are the real parts of the eigenvalues λ_j of the original system's Jocabian. However, if an unstable fixed point is a type-III fixed point ($a_i>1$, $a_j<1$, for some i, j), then it cannot be stabilized by simple AAM at all and, on the other hand, there only exist some special situations such as recursive systems to which the nonuniformly AAM can be applied definitely. The type-IV fixed point (existing at least one j such that either $a_j=1$ or $\lambda_j=1$) is essentially related to bifurcation phenomenon and can be changed into either a type-I or a type-II fixed points through varying the original system's parameters.

Is there an approach that preserves all advantages of the AAM and the YLM method and at the same time removes all disadvantages of them? According to the analytical and numerical investigations of this paper, we show that the answer is positive.

Note that although it is just the discrete time systems that are discussed in this paper, the approach developed here can also be applied to control the flows just by taking the corresponding Poincaré sections.

Now, consider an n-dimensional chaotic discrete system defined by

$$\mathbf{x}_{k+1}=\mathbf{F}(\mathbf{x}_k), \tag{1}$$

where $\mathbf{x}\in\mathbf{R}^n$ is an n-dimensional vectors, and $\mathbf{F}$ is the smooth vector field.

Let $\mathbf{x}_f$ be the fixed point of the system (1), i.e., $\mathbf{x}_f=\mathbf{F}(\mathbf{x}_f)$. In order to stabilize this fixed point, we take the following control strategy described by

$$\mathbf{x}_{k+1}=\mathbf{F}(\mathbf{x}_x)+\mathbf{M}(\mathbf{F}(\mathbf{x}_t)-\mathbf{x}_k), \tag{2}$$

where $\mathbf{M}$ is an $n\times n$ matrix to be determined. Although Eq. (2) takes the form of the AAM, it should be noted that $\mathbf{M}$ is restricted to be a diagonal matrix in the AAM [15], however, it is not always the case in our method. It is easy to know that the systems (1) and (2) share exactly the same set of fixed points as demonstrated in Ref. [15]. Now let us define an infinitesimal deviation of $\mathbf{x}_k$ from $\mathbf{x}_f$ as $\delta\mathbf{x}_k=\mathbf{x}_k-\mathbf{x}_f$. After taking a linear approximation of Eq. (2) in a neighborhood, W, of the fixed point $\mathbf{x}_f$, one gets

$$\delta\mathbf{x}_{k+1}\approx\mathbf{J}\,\delta\mathbf{x}_k+\mathbf{M}(\mathbf{J}-\mathbf{I})\,\delta\mathbf{x}_k, \tag{3}$$

where

 ©2001 The American Physical Society

$$\mathbf{J}=\left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}_k}\right)_{x_k=x_f}$$

is the Jacobian matrix of the original system $\mathbf{F}$ evaluated at the fixed point $\mathbf{x}_f$, and $\mathbf{I}$ is the $n\times n$ identity matrix. In practice, the matrix $\mathbf{J}$ can be experimentally accessible by taking the well known embedding technique [4,16]. The goal of controlling here is to make $\lim_{k\to\infty}|\delta\mathbf{x}_k|\to 0$ (which implies that $\mathbf{x}_k\to\mathbf{x}_f$, as $k\to\infty$). For this aim, we set

$$\delta\mathbf{x}_k=\sigma(k-k_0)\,\delta\mathbf{x}_{k_0}, \tag{4}$$

where $\delta\mathbf{x}_{k_0}=\mathbf{x}_{k_0}-\mathbf{x}_f$, $\mathbf{x}_{k_0}\in W$ is the point from, hence, the control is imposed on the original freely evolved system. Without loss of generality, one may choose $k_0=0$ hereafter. And $\sigma(k)$ is a scalar function of its argument, which satisfies $\sigma(k)\to 0$, as $k\to\infty$. Substituting Eq. (4) into Eq. (3) and eliminating $\delta\mathbf{x}_{k_0}$, we have

$$\mathbf{M}=\left(\frac{\sigma(k+1)}{\sigma(k)}\mathbf{I}-\mathbf{J}\right)(\mathbf{J}-\mathbf{I})^{-1}, \tag{5}$$

where we have assumed that the matrix $(\mathbf{J}-\mathbf{I})$ is invertible in the above.

There are many possible ways to define the function $\sigma(k)$, such as $\sigma(k)=\exp(-\alpha k)$ $(\alpha>0)$ or $\sigma(k)=\beta/k$, etc., here α, β are all constants. In this paper, we would rather define $\sigma(k)$ as

$$\sigma(k)=\gamma^k, \tag{6}$$

where γ is a constant and $\gamma\in(-1,1)$. Making use of Eq. (6), the matrix $\mathbf{M}$ now becomes

$$\mathbf{M}=(\gamma\mathbf{I}-\mathbf{J})(\mathbf{J}-\mathbf{I})^{-1}, \tag{7}$$

where γ is a constant and $\gamma\in(-1,1)$ as mentioned above.

The control method as given by Eq. (7) doesnot require any *a priori* analytical knowledge of the system under investigation, since the elements of matrix $\mathbf{J}$ can be gotten from experimental data by using the known embedding technique. As concerns the size of converging region, i.e., the neighborhood W defined previously, it is also under investigation. In addition, similar to the YLM method, our method is also formulated for an n-dimensional system with n being an integer. So our method can then be applied to any finite-dimensional system in principle, including chaotic and hyperchaotic systems, under the condition that the matrix $(\mathbf{J}-\mathbf{I})$ is invertible. Furthermore, by choosing an appropriate value of γ between -1 and 1, one may have an optimal control through Eq. (7).

As compared to the YLM method, first, our method doesnot require any adjustable controlling parameters in advance, and so it can be applied to much more extensive systems. Second, once the constant γ is chosen in the range of $(-1, 1)$, then $\mathbf{M}$ is definitely determined and need not be changed with the discrete time. Therefore, it is much simple to implement.

On the other hand, by comparison with the AAM, the main progress of our method is that it can be equally applied to different types of the fixed point, particularly, to the ones where AAM cannot be applied at all, as illustrated in the example below.

We have successfully applied the proposed method to several typical chaotic and hyperchaotic systems, such as coupled logistic maps and Hénon maps [17], etc. Here, in order to compare our method with the YLM method and the AAM more definitely, we discuss the following two-dimensional map [14,18] described by

$$x_{k+1}=1-2(x_k^2+y_k^2)+p,$$
$$y_{k+1}=-4x_k y_k+q. \tag{8}$$

To be an illustrative example, this map can be investigated more analytically. In Ref. [14], the parameters p and q are taken to be the adjustable controlling parameters. By adjusting p and q, the unstable orbit is stabilized to the desired fixed point. However, we may take $p=q=0$ in this work. Now, there exist four different fixed points for map (8). They are (0.5, 0.0), (−1.0, 0.0), (−0.25, −0.75), and (−0.25, 0.75), respectively. Here we take the latest one, i.e., the point (−0.25, 0.75), as an application. For the fixed point, the Jacobian matrix is

$$\mathbf{J}=\begin{pmatrix} 1.0 & -3.0 \\ -3.0 & 1.0 \end{pmatrix}.$$

The two eigenvalues of $\mathbf{J}$ are $\lambda_1=-2.0$ and $\lambda_2=4.0$, respectively. It can be verified definitely that this fixed point cannot be stabilized by using the AAM, or the nonuniform AAM. But this is not the case for our method. According to Eq. (7), one knows that

$$\mathbf{M}=\begin{pmatrix} -1.0 & -\dfrac{\gamma-1}{3} \\ -\dfrac{\gamma-1}{3} & -1.0 \end{pmatrix},$$

where γ is a constant and $\gamma\in(-1,1)$. The two eigenvalues of $\mathbf{M}$ are $-1+(\gamma-1)/3$ and $-1-(\gamma-1)/3$, respectively. Since the constant satisfies $\gamma\in(-1,1)$, the eigenvalue $-1-(\gamma-1)/3\in(-1,1)$. That is, one of the unstable directions becomes stable under the control. Then the equation analogous to Eq. (2) is

$$x_{k+1}=1-2(x_k^2+y_k^2)-[1-2(x_k^2+y_k^2)-x_k]-(\gamma-1)$$
$$(-4x_k y_k-y_k)/3,$$
$$y_{k+1}=-4x_k y_k-(\gamma-1)[1-2(x_k^2+y_k^2)-x]/3$$
$$-(-4x_k y_k-y_k). \tag{9}$$

The numerical results are shown in the Fig. 1, for $\gamma=0.5$. In Figs. 1(a) and 1(b), the curves of x_k vs k and y_k vs k are plotted, respectively. Note that the control is imposed on the underlying system at the moment $k_0=0$, as indicated by the

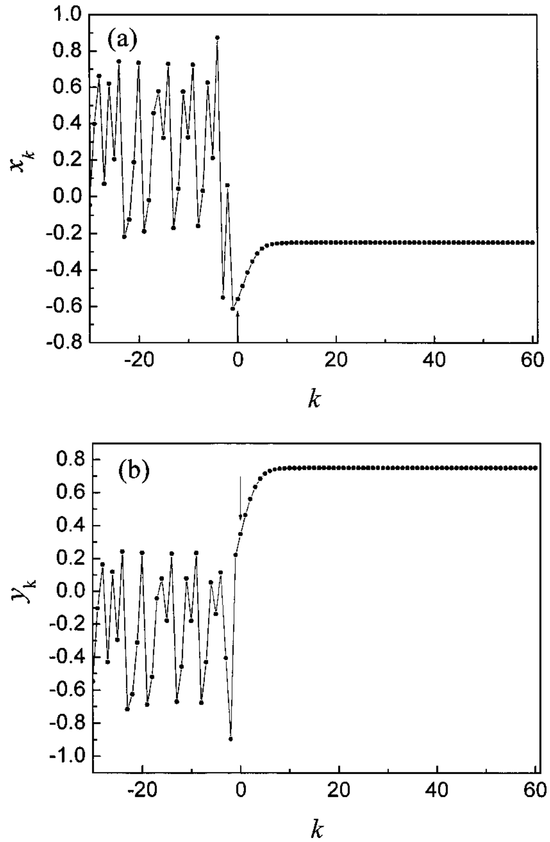

FIG. 1. Numerical results of Eq. (9), for $\gamma=0.5$. (a) and (b) are curves of x_k vs k and y_k vs k, respectively. The iterations are started at an arbitrary initial point and 5000 iterations are omitted as transients. The control is imposed on the system at the moment $k_0=0$ as indicated by the arrow.

arrow in the figure. It is shown that the unstable orbit is stabilized to the desired fixed point monotonically and quickly. In Fig. 2, the converging region W is illustrated schematically. The way that we use to evaluate the size of W is described below. First, let us iterate system (9) for each initial condition $(x_0,y_0)=(r\cos\theta, r\sin\theta)$ (at beginning, $r=0$) and check out whether it is converging. Here, r and θ are polar coordinates (origin at $\mathbf{x}_f$). If $d(=|\mathbf{x}_k-\mathbf{x}_f|)$ is smaller than a given value $d_{\min}$, then one preserves the value of θ unchanged and gradually increases the value of r with step width Δr. Iterate system (9) again until r equals some value r_c where d is larger than another given value $d_{\max}$. This point $(r_c\cos\theta, r_c\sin\theta)$ is just one border point on W.

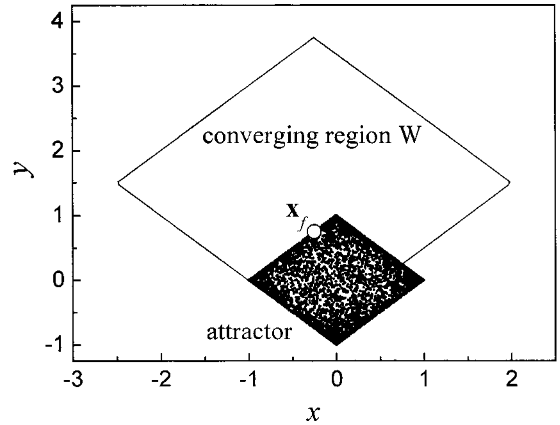

FIG. 2. Schematic illustration of the converging region W for the unstable fixed point $\mathbf{x}_f=(-0.25,0.75)$. Here the attractor of original system is also shown simultaneously for comparison. Starting from any inner point of the converging region, the unstable orbit will be stabilized to $\mathbf{x}_f$ quickly by using the proposed control strategy.

Next, increasing θ with step width $\Delta\theta$ and setting $r=0$, let us repeat the above process again until another border point is found. Here, $\theta\in(0,2\pi)$. In addition, $\Delta r=0.005$, $\Delta\theta=2\pi/500$, $d_{\min}=10^{-4}$, and $d_{\max}=8.0$ are chosen, respectively, in the computations. It is interesting to find that the border of the converging region W looks like a rhomboid. Starting from any inner point of the converging region W, the unstable orbit will approach the fixed point finally by using the proposed control strategy. For comparison, the attractor of the original system is also illustrated in the figure.

In summary, we present a general method for controlling chaotic system, and particularly for hyperchaotic system. The method is based on the variable feedback control and generalizes the AAM. Since it doesn't require *a priori* analytical knowledge of the system under investigation, the method can be conveniently applied to a large class of practical cases. Furthermore, as opposed to the YLM method [14], the OGY method [4] and its variants, the method introduced here needs not any accessible and adjustable controlling parameters. So it is appropriate for much more number of problems.

We acknowledge the financial support of this work by the special funds for the Major State Basic Research Projects of China under Grant No. G2000067104 and the National Pandeng Research Project (95-Yu-41) of China.

[1] D. Auerbach, Phys. Rev. A **41**, 6692 (1990); D. Pierson and F. Moss, Phys. Rev. Lett. **75**, 2124 (1995).
[2] R. L. Davidchack and Y. C. Lai, Phys. Rev. E **60**, 6172 (1999).
[3] P. Schmelcher and F. K. Diakonos, Phys. Rev. Lett. **78**, 4733 (1997).
[4] E. Ott, C. Grebogi, and J. A. York, Phys. Rev. Lett. **64**, 1196 (1990); D. Auerbach, C. Grebogi, E. Ott, and J. A. York, *ibid.* **69**, 3479 (1992).
[5] E. R. Hunt, Phys. Rev. Lett. **67**, 1953 (1991).
[6] J. Güémez and M. A. Matías, Phys. Lett. A **181**, 29 (1993); M. A. Matías and J. Güémez, Phys. Rev. Lett. **72**, 1455 (1994).
[7] B. Peng, V. Petro, and K. Showalter, J. Phys. Chem. **95**, 4957 (1991).
[8] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[9] F. H. Abed, H. O. Wang, and R. C. Chen, Physica D **70**, 154 (1994).
[10] G. Hu, J. Yang, and W. Liu, Phys. Rev. E **58**, 4440 (1998).
[11] K. A. Mirus and J. C. Sprott, Phys. Rev. E **59**, 5313 (1999).

[12] J. Escalona and P. Parmananda, Phys. Rev. E **61**, 5987 (2000).
[13] N. J. Corron, S. D. Pethel, and B. A. Hopper, Phys. Rev. Lett. **84**, 3835 (2000).
[14] L. Yang, Z. Liu, and J. Mao, Phys. Rev. Lett. **84**, 67 (2000).
[15] W. Huang, Phys. Rev. E **61**, R1012 (2000); **62**, 3455 (2000).
[16] F. Takens, in *Dynamical Systems and Turbulence*, edited by D. Rand and L. S. Young (Springer-Verlag, Berlin, 1981), p. 230; J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[17] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).
[18] K. Kaneko, Physica D **34**, 1 (1989).